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Decision Support

Modeling the impact of product quality on dynamic pricing and advertising policies[☆]

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ABSTRACT

The marketing-mix of price–quality and advertising–quality relationship is well studied. Less understood is the price–advertising–quality relationship. This article fills the gap, investigating the interplay between price, advertising, and quality in an optimal control model. Our results generalize the condition of Dorfman–Steiner in a dynamic context. Also, they point to the impact of greater product quality on the dynamic policies of pricing and advertising. Furthermore, a phase diagram analysis shows that quality develops monotonically in time and converges to a unique steady state. We also show that quality investment could either decrease or increase over time but this depends on its effectiveness. Our results spot the profitable opportunities of a firm managing a more complex marketing-mix.

1. Introduction

The marketing-mix of the price–quality and advertising–quality relationships have been extensively studied. Surveys in dynamic pricing (Chen & Chen, 2015; Den Boer, 2015; Elmaghraby & Keskinocak, 2003) and in dynamic advertising (Erickson, 1995; Feichtinger, Hartl, & Sethi, 1994; Huang, Leng, & Liang, 2012; Jørgensen & Zaccour, 2014; Sethi, 1977) reveal the separate analysis of both relationships in prior research. In practice though, recent media claims recall that firms need to manage altogether (opposing two-by-two) price, advertising, and quality. For instance, this marketing-mix concern arises for the American telecom industry with AT&T (Flint, 2019, *The Wall Street Journal*) and also in the vanilla industry of Madagascar (Board, 2019, *The Economist*). Even for the hypothetical market of driverless car, which interests classical automotive groups such as PSA – program Autonomous Vehicle for All – and new entrants like Google – program Waymo, there is a discussion about the simultaneous setting of price, advertising, and quality (Nuttall, 2019, *Financial Times*). Despite its practical relevance, the price–advertising–quality relationship has received little attention by scholars so far, as attested by the influential textbooks of Dockner, Jorgenssen, Long, and Gerhard (2000) and Jørgensen and Zaccour (2012). This article bridges the gap, offering a theoretical foundation of the price–advertising–quality relationship and of the impact of quality on dynamic pricing and advertising policies.

In this article, we investigate the joint dynamic pricing, advertising, and quality investment policies, focusing on the conditions under which greater quality drives higher or lower price and advertisement. We propose an optimal control model coping with the following elements: a firm sets pricing, advertising, and quality investment policies over time. Consumers are sensitive to price, advertisement and product quality; quality is costly to produce for the firm. The preferences of consumers and the organization of the firm are tied to the dynamics of demand and supply, which, in turn, are linked to the dynamic pricing, advertising, and quality investment policies. Literature on marketing-mix with dynamic pricing, advertising, and quality thus informs this research.

The closest articles to our research are Chenavaz (2017a) and Chenavaz and Jasimuddin (2017), which consider the impact of quality on price and on advertising, respectively. Considering jointly the price–quality and advertising–quality relationships, we show how advertisement affects the price–quality relationship and how price changes the advertising–quality relationship. In their vein, we offer structural (opposing parametric) results, enabling to solve the monopolist's dynamic behavior problem under the most

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general demand formulation, which accounts for nonlinearities and dynamics in response to quality changes.

Our main results are as follows. First, we generalize the condition of Dorfman and Steiner (1954) in a dynamic context, stressing the profit-maximizing conditions for the price–advertising, price–quality, and advertising–quality relationships. Second, we develop the pricing–advertising–quality relationship, examining the conditions under which better quality triggers higher or lower price and advertisement. Third, a phase diagram analysis learns that quality develops monotonically in time and converges to a unique steady state. We also show that quality investment could either decrease or increase over time but this depends on its effectiveness. Such results foster the understanding of more complex marketing-mix opportunities, enhancing the profitability of the firm.

The paper is organized as follows. Section 2 presents the related contributions. Section 3 develops the model, whereas Section 4 generalized the rules of Dorfman–Steiner to a dynamic context. Section 5 studies the impact of quality on dynamic pricing and advertising. Section 6 computes the optimal trajectories using a phase plane analysis, and Section 7 concludes.

2. Related contributions

Dynamic marketing-mix research provide numerous related contributions. Such contributions focusing on dynamic pricing, advertising, and product quality, are regularly surveyed. Some of these surveys focus on pricing (Dockner et al., 2000; Jørgensen & Zaccour, 2012; Den Boer, 2015) and others on advertising (Feichtinger et al., 1994; Bagwell, 2007; Huang et al., 2012; Jørgensen & Zaccour, 2014). More specific literature pointers have been provided recently for the price–quality relationship (Chenavaz 2017a; Vörös, 2019; Ni & Li, 2019) and for the advertising–quality relationship (Chenavaz & Jasimuddin, 2017).

We now briefly review research using optimal control, which provide elements about the relationships between price, quality, and advertising. Table 1 presents in chronological order some of the main optimal control models discussed below. The table helps to understand the distinguishing mathematical formulation and managerial interests in the literature, and thus the positioning of our research.

Price and advertising are examined together by Piga (2000) with a model of sticky prices with advertising, Helmes, Schlosser, and Weber (2013) with oligopolistic strategies, and Schlosser (2017) the stochastic element at the demand side. An entertainment event is investigated by Jørgensen, Kort, and Zaccour (2009), a marketing channel by Amrouche, Martín-Herrán, and Zaccour (2008), and product diffusion by Helmes and Schlosser (2015).

Price and quality are jointly studied as follows: Vörös (2006), Chenavaz (2011, 2012), and Vörös (2013) consider improvement in productivity. Teng and Thompson (1996) and Mukhopadhyay and Kouvelis (1997) look at the joint dynamic pricing and quality policies, where quality is chosen by the firm. Reference effects are considered by Gavious and Lowengart (2012), Xue, Zhang, Tang, and Dai (2017), Chenavaz and Paraschiv (2018), and product returns by De Giovanni and Zaccour (2020). The conditions determining when a product better quality is more or less expensive are studied in Chenavaz (2017a), and generalized through the introduction of goodwill by Ni and Li (2019) and through the salvage value by Vörös (2019). The joint effects of price and quality investment are also considered when there is a potential competitor (Ha, Long, & Nasiry, 2015), when there exists risk-averse attitude (Xie, Yue, Wang, & Lai, 2011), and with the addition of a new channel in a supply chain (Chen, Liang, Yao, & Sun, 2017). Eventually, Karaer and Erhun (2015) and Cui (2019) analyze the role of product quality and pricing in preventing market entrant.

Advertising and quality are jointly analyzed in the following contributions. Colombo and Lambertini (2003) look at product differentiation. El Ouardighi, Feichtinger, Grass, Hartl, and Kort (2016a,b) focus the role of word of mouth. More recently, Chenavaz and Jasimuddin (2017) investigate when a product of better quality increases or decreases advertising. In this last research, and close to our contribution, price is assumed to be given by an inverse demand function (see Table 1), that is price is not directly controlled by the firm. In practice though, the firm, which differentiates its product by leveraging the quality and advertising levels, has also some freedom in price setting. There it has to take into account that setting the price in turn affects the advertising–quality relationship (Chenavaz & Jasimuddin, 2017), which generalize to a price–advertising–quality relationship (this article). Also, from a conceptual point of view, Chenavaz and Jasimuddin (2017) proves Tellis and Fornell's (1988) conjecture about the advertising–quality relationship, whereas this article generalizes Dorfman and Steiner's (1954) condition to a dynamic situation.

Only few contributions combine pricing, advertising, and quality, opposing tow by two. In this respect Fruchter (2009) looks at decisions of price and advertising with perceived quality, introducing a psychological element. Caulkins et al. (2015) analyze history dependence with experience quality. Recently, Ni and Li (2019) consider the carry-over effect role of goodwill in the price–quality relationship, without investigating the price–advertising–quality relationship. Also, as it appears in Table 1, most approaches remain parametric, offering stronger, but less general results.

To the best of our knowledge, no research examine *both* the price–advertising–quality relationship explicitly, that is with price, advertising, and quality being all control or state variables *and* with general (structural) formulations for both the demand function and state dynamics, imposing only little restriction on the relationships among the variables. In this article, we bridge the gap providing an optimal control model, which simultaneously address these two points. Our results offer a deeper understanding of the price–advertising–quality relationship in a general framework.

3. Model formulation

3.1. Model development

The intertemporal behavior of a monopolist is modeled in an optimal control setting. The planning horizon is infinite and the time $t \in [0, \infty)$ is continuous.

3.1.1. Quality

At each time t, the firm chooses the level of quality investment $u(t) \in \mathbb{R}_+$ that improves product quality $q(t) \in \mathbb{R}_+$. Thus, investment in quality $u(t)$ is a decision (or control) variable and quality $q(t)$ is a state variable. In other words, quality is modeled as a system state.

The quality dynamics evolve according to

$$\dot{q}(t) = K(u(t), q(t)) \text{ with } q(0) = q_0, \tag{1}$$

where $K : \mathbb{R}_+^2 \to \mathbb{R}_+$ is twice continuously differentiable.[1] The integration of (1) yields the cumulative level of quality $q(t) = q_0 + \int_0^t K(u(s), q(s))ds$. To simplify presentation, we shall omit the arguments from the functions whenever there is no confusion, especially the temporal argument t.

[1] The notations $\dot{z}$ and z_x state for the time derivative of z and the first order derivative of z with respect to x; the notations z_{xx} and z_{xy} denote the second order derivative of z with respect to x and the cross derivative of z with respect to x and y.

Table 1
Selected optimal control models of price, advertising, and quality.

References	System dynamics	Main contributions
Teng and Thompson (1996)	$\dot{s} = S(p, q, s)$	Show the different possible relationships between price p and quality q
Piga (2000)	$\dot{p} = s[\alpha + \beta \sum a_j - \sum Q_j - p]$	Integration of advertising in a sticky price duopoly
Fruchter (2009)	$\dot{q} = \alpha p + \beta a - \delta q$	Focus on the impact of perceived quality and the pricing and advertising policies
Caulkins et al. (2011)	$\dot{G} = \alpha(\beta p - G)$	Investigate the pricing policy for conspicuous goods when goodwill exerts influence
Gavious and Lowengart (2012)	$\dot{r}_q = \beta(q - r_q)$	Role of reference quality in the pricing and quality policies
Chenavaz (2012)	$\dot{q} = Q(u_q, q)$ $\dot{c} = C(u_c, c)$	Show the impact of production cost and product quality on the pricing policy
Caulkins et al. (2015)	$\dot{M} = a/M^{\theta}$ $-\delta(M - \alpha p)(q_{max} - q) - \beta M$	Role of market potential in the marketing-mix policy of price, advertising, and quality
Feng, Zhang, and Tang (2015)	$\dot{g} = a - \delta g$ $\dot{i} = -(\alpha - \beta p + \gamma g) - \delta i$	Role of goodwill and inventory dynamics in the marketing-mix policy of price and advertising
El Ouardighi et al. (2016a)	$\dot{p} = c$ $\dot{s} = \alpha[\beta + a]s[(\gamma - \theta s) - p] - \delta s$	Role of costly price adjustment on sales and advertising policies
El Ouardighi, Feichtinger, Grass, Hartl, and Kort (2016b)	$\dot{q} = u(1 - q)$ $\dot{s} = \{[(\alpha + \beta a)q - \beta(1 - q)](1 - s/M)$ $-[1 - (1 - \delta q)]\}s$	Role of quality improvement and advertising policies investment on product diffusion
Pan and Li (2016)	$\dot{q} = u_q - \delta_q q$ $\dot{c} = u_c - \delta_c c$	Role of process-product innovation on the pricing policy
Chenavaz (2017a)	$\dot{q} = Q(u, q)$	Show why a product of better quality may be less expensive
Chenavaz and Jasimuddin (2017)	$\dot{q} = Q(u, q)$	Show when advertising increases or decreases with better product quality
Chenavaz and Paraschiv (2018)	$\dot{r} = \beta(p - r)$ $\dot{i} = -D(p, r)$	Show when the selling price increases or decreases with the reference price for a general demand function
Vörös (2019)	$\dot{q} = \alpha u$	Differentiate between strategic and non-strategic quality in the price–quality relationship
Ni and Li (2019)	$\dot{q} = Q(u, q)$ $\dot{g} = G(a, q, g)$	Role of advertising policy and goodwill in the price–quality relationship

Notes. We use the following unified notations for the variables: a, advertising; c, production cost; D, demand; g, goodwill; i, inventory; M, market potential; q, product quality; Q, output; r, reference price; s, sales; u, investment. Also, j is a firm index and α, β, γ, δ are positive parameters.

Investment in quality u increases quality q with diminishing returns and quality decreases autonomously, capturing the aging of technology.

$$K_u > 0,\ K_{uu} \leqslant 0,\ K_q < 0. \tag{2}$$

These assumptions encompass the parametric instances $\dot{q} = u - \delta q$ and $\dot{q} = \sqrt{u} - \delta q$, with the constant rate of decay $\delta > 0$, as in Jørgensen and Zaccour (2012) and Xue et al. (2017).

3.1.2. Cost

The unitary production cost function $C : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable and increases with quality q. Therefore the cost is $C = C(q)$ with

$$C_q \geqslant 0. \tag{3}$$

A similar cost formulation is used in Xue et al. (2017) and De Giovanni and Zaccour (2020). The independence of cost to quality $C_q = 0$ and the increase of cost with quality $C_q > 0$ describe, for example, the software and hardware industries (Chenavaz, 2017a).

3.1.3. Demand

The firm decides at each time t the price level $p(t) \in \mathbb{R}_+$ and the advertising expense $a(t) \in \mathbb{R}_+$. Quality is defined here as a *search* (or *design*) attribute, which is easily knowable by search and for which consumers prefer more than less. The demand function $D : \mathbb{R}_+^3 \to \mathbb{R}_+$ is twice continuously differentiable. The demand D depends jointly on price p, advertising a, and quality q, that is $D = D(p, a, q)$.

Demand reduces with price. Demand rises with advertising with diminishing returns. Demand increases with product quality. It is more difficult to increase demand with more advertising and better quality when price is high. There is a synergy phenomenon between advertising and quality, that is, the impact of advertising (quality) on demand is greater with greater quality (advertising).

$$D_p < 0,\ D_a > 0,\ D_{aa} \leqslant 0,\ D_q > 0,\ D_{pa} \leqslant 0,\ D_{pq} \leqslant 0,\ D_{aq} \geqslant 0. \tag{4}$$

This general demand function places little restriction on the way price, advertising, and quality affects demand. Indeed, this demand function is compatible with the persuasive and informative views (Bagwell, 2007; Chenavaz & Jasimuddin, 2017). The persuasive view posits that advertising changes consumer preferences; the informative view supposes that advertising provides product information. In each case, greater advertising boosts demand.

Table 2
Notations.

t	= time,
r	= interest rate,
$p(t)$	= unit price at time t (control variable),
$a(t)$	= advertising expense at time t (control variable),
$u(t)$	= quality investment at time t (control variable),
$q(t)$	= product quality at time t (state variable),
$\dot{q}(t)$	= $dq(t)/dt = K(u, q)$ = quality dynamics at time t,
$\lambda(t)$	= current-value adjoint variable at time t,
$C(q)$	= unit production cost,
$D(p, a, q)$	= demand,
$\pi(p, a, u, q)$	= $[p - C(q)]D(p, a, q) - a - u$ = current profit,
$H(p, a, u, q, \lambda)$	= current-value Hamiltonian

3.2. Model analysis

Table 2 defines the notations used in the model analysis.

The current profit π with values in $\mathbb{R}$ writes

$$\pi(p, a, u, q) = [p - C(q)]D(p, a, q) - a - u. \tag{5}$$

It appears now clearly that quality and advertising share similar features, both stimulating demand and implying a cost. The essential differences are that (1) advertising directly implies a fixed cost

a whereas quality indirectly creates a fixed cost via quality investment u and (2) quality also generates a variable cost $C(q)$.

The firm maximizes the intertemporal profit (or total present value of profit) by simultaneously finding the optimal trajectories of pricing, advertising, and quality investment over the planning horizon. The firm accounts for the quality dynamics and the discount rate $r \in \mathbb{R}_+$. Formally, the objective function of the firm reads

$$\max_{p,a,u} \int_0^\infty e^{-rt}\pi(p,a,u,q)dt, \tag{6}$$

$$\text{subject to } \dot{q} = K(u,q), \text{ with } q(0) = q_0. \tag{7}$$

The intertemporal profit maximization problem is solved with the necessary and sufficient optimality conditions of Pontryagin's maximum principle. On this basis, the shadow price (or current-value adjoint variable) $\lambda(t)$ represents the marginal value of quality on the intertemporal profit at t, and the current-value Hamiltonian H writes

$$H(p,a,u,q,\lambda) = [p - C(q)]D(p,a,q) - a - u + \lambda K(u,q).$$

The current-value Hamiltonian H sums the current profit $(p - C)D - a - u$ and the future profit λK. As such, H measures the intertemporal profit.

The maximum principle implies the dynamic of the shadow price λ:

$$\dot{\lambda} = r\lambda - H_q = r\lambda - [-C_q D + (p - C)D_q + \lambda K_q], \tag{8}$$

with the transversality condition for a free terminal state and infinite terminal time $\lim_{t\to\infty} e^{-rt}\lambda(t) = 0$.

The intertemporal value at time t of a marginal increase in quality q is given by the integration of (8) with the transversality condition, computing

$$\lambda(t) = \int_t^\infty e^{-\left(r - \int K_q d\mu\right)(\tau - t)}[(p - C)D_q - C_q D]d\tau, \tag{9}$$

where we abuse the notation by denoting $\int K_q d\mu$ for $\int_{\tau-t}^{\infty} K_q(u(\mu), q(\mu))d\mu$.

Assuming the existence of an interior solution for advertising and investment in quality, the monopolist maximizes the intertemporal profit H if and only if p, a, and u satisfy the necessary first-order conditions:

$$H_p = 0 \Rightarrow D + (p - C)D_p = 0, \tag{10a}$$

$$H_a = 0 \Rightarrow (p - C)D_a - 1 = 0, \tag{10b}$$

$$H_u = 0 \Rightarrow K_u - \frac{1}{\lambda} = 0. \tag{10c}$$

Asterisk denoting optimality, let $p^*(a, u)$ be the price that verifies (10a). This price maximizes the intertemporal profit of any levels of advertising and quality investment. Similarly, $a^*(p, u)$ and $u^*(p, a)$ denote the advertising rate satisfying (10b) and (10c); they maximize the intertemporal profit for any level of price and quality investment and for any level of price and advertising. The maximum of the intertemporal profit is achieved when the firm chooses together a price, advertising, and quality investment triple such that $(p^*, a^*, u^*) = (p^*(a^*, u^*), a^*(p^*, u^*), u^*(p^*, a^*))$. We omit now the asterisk notation, all equations referring to the optimal solution if not otherwise stated.

For the necessary first order conditions on H (10a)–(10c) to yield a maximizing solution, assuming its existence, a sufficient second order condition is the concavity of H. H is concave if and only if the Hessian matrix is semi-negative definite, that is[2]

$$H_{pp} \leqslant 0, \tag{11a}$$

$$\begin{vmatrix} H_{pp} & H_{pa} \\ H_{ap} & H_{aa} \end{vmatrix} \geqslant 0, \tag{11b}$$

$$\begin{vmatrix} H_{pp} & H_{pa} & H_{pu} \\ H_{ap} & H_{aa} & H_{au} \\ H_{up} & H_{ua} & H_{uu} \end{vmatrix} \leqslant 0, \tag{11c}$$

with

$$H_{pp} = 2D_p - (p - C)D_{pp}, \tag{12a}$$

$$H_{aa} = (p - C)D_{aa}, \tag{12b}$$

$$H_{uu} = \lambda K_{uu}, \tag{12c}$$

$$H_{pa} = H_{ap} = D_a + (p - C)D_{pa}, \tag{12d}$$

$$H_{pu} = H_{up} = 0, \tag{12e}$$

$$H_{au} = H_{ua} = 0. \tag{12f}$$

Recall that H builds on functions all assumed to be twice continuously differentiable. Therefore, all partial derivatives of H are themselves differentiable, and Schwarz's theorem applies ($H_{ij} = H_{ji}$ for all $i, j = p, a, u, q$).

Eq. (12e) states that in the intertemporal profit H there are no interaction effects between price and investment in quality. Similarly, Eq. (12f) implies that with respect to H there are also no interaction effects between advertising and quality investment.

Substituting (10a) into (12a) and recalling (11a) implies

$$2 - D\frac{D_{pp}}{D_p^2} \geqslant 0, \tag{13}$$

which dictates that D cannot be too convex in p in the sense that the second order derivative of D with respect to p cannot be too large. Such an assumption has been widely used in the literature (see for example Kalish, 1983; Dockner et al., 2000; Vörös, 2006; 2019;Chenavaz, 2012; 2017b; Jørgensen and Zaccour, 2012; and Ni & Li, 2019).

Substituting (10b) into (12b) and recalling (11a) and (11b) implies

$$-\frac{D_{aa}}{D_a^2} \geqslant 0, \tag{14}$$

which is verified because of (4).

Conditions (11a)–(11c) together imply that $\lambda K_{uu} \leq 0$, which, together with (2), leads to

$$\lambda(t) \geqslant 0, \quad \forall t \in [0, \infty), \tag{15}$$

meaning that greater quality always increases the intertemporal profit.

4. Generalizing the rules of Dorfman–Steiner

In this section, we generalize the static rules of Dorfman and Steiner (1954) in a dynamic context, looking at the dynamic price–advertising–quality relationship. The rules of Dorfman and Steiner (1954) have to be verified for levels of price and advertising to be candidate as optimal values. As such, these rules provide a useful benchmark against which potential price and advertising levels can be evaluated.

[2] If all inequalities hold strictly, the Hessian matrix is strictly negative definite, H is strictly concave, and the solution is unique.

We first posit elasticity notations. Let $\eta_p \equiv -D_p \frac{p}{D}$ be the price elasticity of demand, $\eta_a \equiv D_a \frac{a}{D}$ the advertising elasticity of demand, and $\eta_q \equiv D_q \frac{q}{D}$ the quality elasticity of demand.

Conditions (10a) and (10b) imply $p - C = -\frac{D}{D_p}$ and $p - C = \frac{1}{D_a}$. From these equalities and (9), we obtain that

$$DD_a + D_p = 0, \tag{16a}$$

$$\lambda(t) = \int_t^{\infty} e^{-(r-K_q)(\tau-t)} D\left(\frac{\eta_q}{\eta_p}\frac{p}{q} - \frac{dC}{dq}\right)d\tau, \tag{16b}$$

$$\lambda(t) = \int_t^{\infty} e^{-(r-K_q)(\tau-t)} \left(\frac{\eta_q}{\eta_a}\frac{a}{q} - \frac{dC}{dq}D\right)d\tau. \tag{16c}$$

Eq. (16a) assembles the first-order conditions on price and advertising. Eqs. (16b) and (16c) measure the future profit at t of an additional unit of quality in terms of (1) price and quality effects and (2) advertising and quality effects.

4.1. Dorfman–Steiner's rules if future quality matters

Proposition 1. *For a general demand function $D = D(p, a, q)$, the price–advertising, price–quality, and advertising–quality relationships are characterized by*

$$\frac{a}{pD} = \frac{\eta_a}{\eta_p}, \tag{17a}$$

$$\frac{\eta_q}{\eta_p}\frac{p}{q} \geqslant \frac{dC}{dq}, \tag{17b}$$

$$\frac{\eta_q}{\eta_a}\frac{a}{q} \geqslant \frac{dC}{dq}D, \tag{17c}$$

in which (17a) is a necessary and sufficient condition and (17b)–(17c) are not necessary but sufficient conditions.

Proof. See Appendix B.1. □

Equality (17a) represents what is well-known as the *condition* of Dorfman and Steiner (1954). This condition stipulates that price and advertising are such that the ratio of advertising to revenue equals the ratio of elasticities of demand with respect to advertising and price. Inequalities (17b) and (17c) represent sufficient conditions. Inequality (17b) states that price and quality are such that the increase in the price that consumers are willing to pay after a quality increase exceeds the unit increase in the cost of quality. Inequality (17c) exposes that advertising and quality are such that the adjustment in advertising following a quality increase outweighs the total increase in the cost of quality. For exhaustiveness, we derive the three conditions of Dorfman and Steiner (1954) and compare them with our results in Appendix A.

The classical condition of Dorfman–Steiner (17a) is robust in our dynamic setting where the firm considers future quality. We revisit now this condition by explicitly showing the dependence of the dynamics of price, quality, and advertising.

4.2. Dorfman–Steiner's condition in dynamics

The condition of Dorfman–Steiner in (17a) provides the static pricing–advertising condition. Proposition 1 shows that this condition holds in a dynamic setting in which the future impact of quality is considered by the firm. Note that this static condition does not directly accounts for product quality. Though this condition must hold during the whole planning period, on which the firm has an optimal behavior. At the optimum, if quality changes, then marginal revenue variations balance marginal cost variations. Such variations in quality also generate variations in pricing and advertising. The link between the dynamics of pricing and advertising on the one side and quality on the other side becomes explicit with the following proposition:

Proposition 2. *For a general demand function $D = D(p, a, q)$, the dynamics of the condition of Dorfman–Steiner (17a) is characterized by*

$$\dot{p}\left(\underbrace{-D_pD_a}_{+}\underbrace{-DD_{ap}}_{+}\underbrace{-D_{pp}}_{\pm}\right) + \dot{a}\left(\underbrace{-D_a^2}_{-}\underbrace{-DD_{aa}}_{+}\underbrace{-D_{pa}}_{+}\right)$$
$$= \dot{q}\left(\underbrace{D_aD_q}_{+}\underbrace{+DD_{aq}}_{+}\underbrace{+D_{pq}}_{-}\right). \tag{18}$$

Proof. See Appendix B.2. □

Proposition 2 quantifies the link between the dynamics of price and advertising and the dynamics of quality when the condition of Dorfman–Steiner (17a) is verified. In other words, Proposition 2 stresses the role of quality within the condition of Dorfman–Steiner, which focuses on the pricing–advertising relationship.

Eq. (18) is undetermined as the system state, quality, is known, and the controls, price and advertising, are unknown. Still, this proposition provides some insights on the consequences of quality variation on the level of price and advertisement. It shows that quality impacts both pricing and advertising in an additive separable manner and it measures the linkage between the impacts on price and advertising.

The signs of the three parentheses in Proposition 2 are unknown. Therefore, when quality increases, price and advertising may both increase or decrease together, or the one may increase while the other decreases. If advertising is constant, price may increase or decrease after a quality increase. Similarly, if price is constant, advertising may raise or fall following better quality.

5. Quality impact on dynamic pricing and advertising

In this section we examine the impact of better quality on price and advertising policies, when the rules of Dorfman and Steiner (1954) studied in the previous section apply. The optimal pricing and advertising policies have to hold at any time of the planning period. Thus, we consider the time derivative of (10a) and (10b). Rearranging terms offers

$$\dot{p}[2D_p + (p - C)D_{pp}] + \dot{a}[D_a + (p - C)D_{pa}]$$
$$= \dot{q}[-D_q - (p - C)D_{pq} + C_qD_p], \tag{19a}$$

$$\dot{p}[D_a + (p - C)D_{ap}] + \dot{a}[(p - C)D_{aa}]$$
$$= \dot{q}[-(p - C)D_{aq} + C_qD_a]. \tag{19b}$$

Recall H_{pp}, H_{aa}, and H_{pa} from (12a), (12b), and (12d) and observe $H_{pq} = -C_qD_p + (p - C)D_{pq} + D_q$ and $H_{aq} = -C_qD_a + (p - C)D_{aq}$. By identification, the precedent equations synthesize to

$$\dot{p}H_{pp} + \dot{a}H_{pa} = -\dot{q}H_{pq}, \tag{20a}$$

$$\dot{p}H_{ap} + \dot{a}H_{aa} = -\dot{q}H_{aq}. \tag{20b}$$

Eqs. (19a) and (19b) express the impact of quality on price and advertising at the highest structural level. The dynamics of the control variables p and a appear on the left hand-side and the dynamics of the state variable q show on the right hand-side. Further, investment in quality u plays no direct role in the dynamics

Table 3
Synthesized marketing-mix implications of Proposition 3.

Case	Condition	Implication
1	$-H_{kq}H_{ll} + H_{lq}H_{kl} > 0$	sign $\dot{k}$ = sign $\dot{q}$
2	$-H_{kq}H_{ll} + H_{lq}H_{kl} < 0$	sign $\dot{k}$ = − sign $\dot{q}$
3	$-H_{kq}H_{ll} + H_{lq}H_{kl} = 0$	sign $\dot{k}$ = unknown

Notes. Indexes $k, l = p, a$ and $k \neq l$.

of price and advertising; investment plays only an indirect role via quality determination, as imposed by (10c).

Recall $H_{ap} = H_{pa}$ and define $H_2 = H_{pp}H_{aa} - (H_{pa})^2$. Note $H_2 \geq 0$ because of (11b). If $H_2 = 0$, then the dynamics of p and a are unknown. If $H_2 > 0$, then the dynamics of p and a are characterized as follows.

Proposition 3. *For a general demand function $D = D(p, a, q)$, the impact of increased quality on the dynamics of price and advertising is given by*

$$\dot{p} = \frac{-H_{pq}H_{aa} + H_{aq}H_{pa}}{H_2}\dot{q}, \; \dot{a} = \frac{-H_{aq}H_{pp} + H_{pq}H_{ap}}{H_2}\dot{q},$$

where $H_2 > 0$.

Proof. See Appendix B.3. □

The dynamic pricing and advertising policies given by Proposition 3 are symmetrical since $H_{ap} = H_{pa}$. Therefore, a synthetic rule writes

$$\dot{k} = \frac{-H_{kq}H_{ll} + H_{lq}H_{kl}}{H_2}\dot{q} \text{ with } k, l = p, a \text{ and } k \neq l,$$

for which the marketing-mix implications are synthesized in Table 3.

Results in Proposition 3 synthesized in Table 3 have the following interpretations. On the one hand, the firm is better off when price and advertising expense increase with quality, provided $-H_{kq}H_{ll} + H_{lq}H_{kl} > 0$ with $k, l = p, a$ and $k \neq l$ (Case 1). In Case 1, price and advertising are complementary to quality. On the other hand, if $-H_{kq}H_{ll} + H_{lq}H_{kl} < 0$ (Case 2), then the firm makes higher profit by decreasing price and advertising effort after a quality increase. In Case 2, price and advertising are substitute to quality. Finally, if $-H_{kq}H_{ll} + H_{lq}H_{kl} = 0$ (Case 3), the firm adopts pricing and advertising schemes for which the evolution is not tied to quality dynamics. In Case 3, price and advertising are independent from quality.

Further, Proposition 3 implicitly posits that the dynamics of price and the dynamics of advertising may not be inferred from each other. In other words, knowledge of price or advertising dynamics constitutes no signal for advertising or price dynamics respectively.

6. Computation of the optimal trajectories

We start out with analyzing our deterministic model. Afterwards, we provide an extension where the horizon date, T, is finite and stochastic.

6.1. Analysis of the deterministic model

We analyze the model (6) and (7) presented in Section 2 with profit function (5), namely

$$\max_{p,a,u} \int_0^\infty e^{-rt}[(p - C(q))D(p, a, q) - a - u]dt, \tag{21}$$

$$\text{subject to} \quad \dot{q} = K(u, q), \text{ with } q(0) = q_0, \tag{22}$$

employing a two-step approach since the “static” controls a and p, do not enter the state dynamics.

In Step 1 we maximize the integrand

$$\pi(p, a, u, q) = (p - C(q))D(p, a, q) - a - u$$

with respect to the “static” controls a and p, yielding an optimal profit function

$$\pi^*(u, q) = \max_{a,p} \pi(p, a, u, q),$$

and optimal control functions $p(q)$ and $a(q)$. Then, in Step 2, we solve an optimal control problem with state variable q and one control variable u. This optimal control problem can be expressed as

$$\max_u \int_0^\infty e^{-rt}[(p(q) - C(q))D(p(q), a(q), q) - a(q) - u]dt,$$
$$\text{subject to} \quad \dot{q} = K(u, q), \text{ with } q(0) = q_0,$$

In order to solve this problem, we use the following specifications. First, we employ the linear demand function

$$D(p, a, q) = \gamma q + \theta\sqrt{a} - \beta p. \tag{23}$$

We also assume that $C(q)$ is linear, i.e.,

$$C(q) = cq. \tag{24}$$

Furthermore, we impose that

$$K(u, q) = q^{-\alpha}\sqrt{u} - \delta q, \quad \alpha > 0. \tag{25}$$

The motivation for expression (25) is as follows: first, note that $\alpha > 0$ reflects the fact that it is more difficult to increase quality if quality is already high. Second, we introduce the term $\sqrt{u}$, because now the control variable u enters the problem in a nonlinear and concave way.

The next proposition summarizes the results of Step 1.

Proposition 4. *Under the condition*

$$\beta \geq \frac{\theta^2}{4}, \tag{26}$$

price and advertising depend on quality in the following way:

$$p(q) = \frac{(2\beta - \theta^2)c + 2\gamma}{4\beta - \theta^2}q, \tag{27}$$

$$a(q) = \left(\frac{\theta(\gamma - \beta c)}{4\beta - \theta^2}\right)^2 q^2. \tag{28}$$

Price increases in quality, if

$$\beta > \frac{\theta^2}{2} - \frac{\gamma}{c}, \tag{29}$$

whereas advertising always increases with quality.
Demand is proportional to quality:

$$D = \frac{2\beta}{4\beta - \theta^2}(\gamma - \beta c)q,$$

which is positive, if

$$\gamma - \beta c > 0. \tag{30}$$

Proof. See Appendix C.1. □

Having determined the functions forms of $p(q)$ and $a(q)$, we are ready to analyze the optimal control problem of Step 2. After some straightforward calculations, the Step 2 problem becomes

$$\max_u \int_0^\infty e^{-rt}[\Pi(q) - u]dt, \tag{31}$$

$$\text{subject to} \quad \dot{q} = q^{-\alpha}\sqrt{u} - \delta q, \tag{32}$$

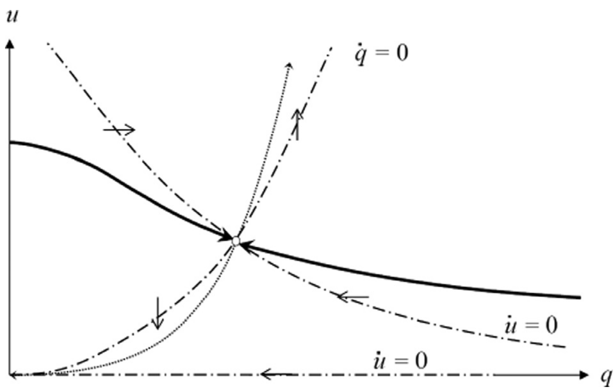

Fig. 1. Downward sloping saddle point path for $\alpha > 1$.

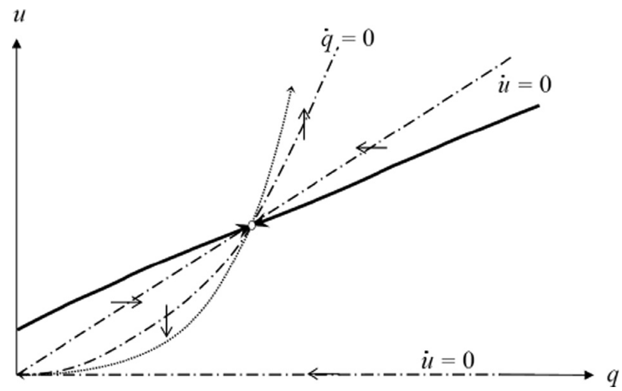

Fig. 2. Upward sloping saddle point path for $\alpha < 1$.

with

$$\Pi(q) = \frac{(\gamma - \beta c)^2}{4\beta - \theta^2} q^2.$$

In Appendix C.2, we use the necessary optimality conditions for this optimal control problem to obtain the canonical system in the state-control plane:

$$\dot{q} = q^{-\alpha}\sqrt{u} - \delta q,$$

$$\dot{u} = 2u(r + \delta + \alpha\delta) - q^{1-\alpha}\frac{2(\gamma - \beta c)^2}{4\beta - \theta^2}\sqrt{u}.$$

It follows that there are two steady states, namely $q = u = 0$, and a unique positive one:

$$\hat{q} = \left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^{\frac{1}{2\alpha}}, \tag{33}$$

$$\hat{u} = \delta^2\left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^{\frac{1+\alpha}{\alpha}}. \tag{34}$$

In Appendix C.3, we investigate the stability properties of the interior steady state (33) and (34), where we conclude that it is a saddle point.

The two isoclines (relevant for the interior equilibrium) are

$$\dot{q} = 0: \quad u = (\delta q)^2, \tag{35}$$

$$\dot{u} = 0: \quad u = \left(\frac{(\gamma - \beta c)^2}{(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^2 q^{2(1-\alpha)}. \tag{36}$$

For $0 < \alpha < 1$ the $\dot{u} = 0$ isocline is upward sloping in the (q, u) plane, while it is downward sloping for $\alpha > 1$. In the hairline case $\alpha = 1$ it is horizontal.

In Fig. 1 we depict the case of a large $\alpha > 1$, where the saddle point path is downward sloping just as the $\dot{u} = 0$ isocline. For low quality, high quality investments are undertaken in order to improve it quickly.

In Fig. 2 we show the case of a small $\alpha < 1$, where the saddle point path is upward sloping and so is the $\dot{u} = 0$ isocline. For low quality, also low quality investments are undertaken and quality increases anyway.

Fig. 3 represents the hairline case of $\alpha = 1$, where the saddle point path is horizontal just like the $\dot{u} = 0$ isocline. Quality investments are constant and do not depend on the current level of quality.

To explain the qualitative differences between the three solutions, consider Fig. 4. From this figure we conclude that for $\alpha > 1$,

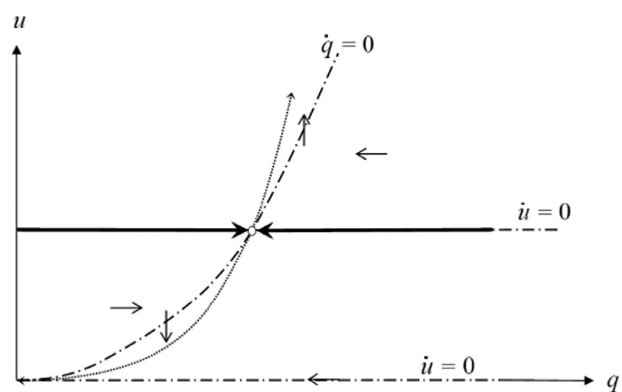

Fig. 3. Horizontal for $\alpha = 1$.

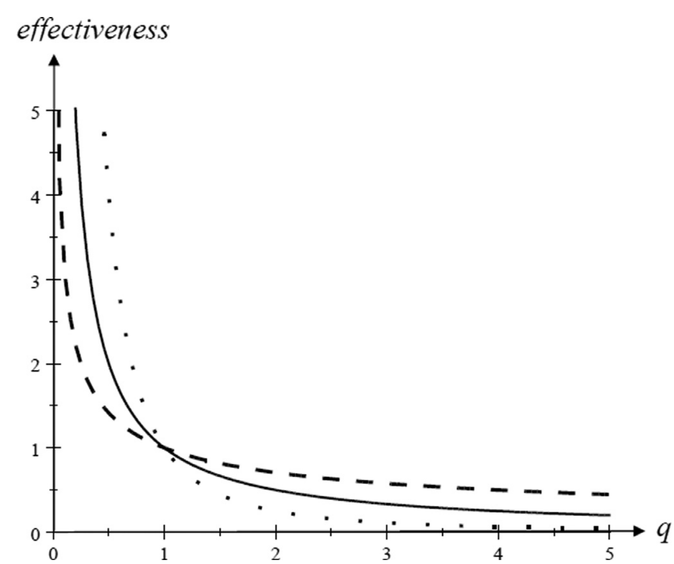

Fig. 4. The effectiveness of quality investment $q^{-\alpha}$, for $\alpha < 1$ (dashed), $\alpha = 1$ (solid), and $\alpha > 1$ (dotted).

quality investment is most effective when quality is small. This explains that quality investments decrease when quality goes up in Fig. 1. For $\alpha < 1$, however, quality investment is relatively more effective when quality is large. This explains that quality investments increase when quality goes up in Fig. 2.

6.2. Stochastic extension

Up to now, we had an infinite horizon. However, we are well aware of the fact that the life cycle of a product is usually finite.

Moreover, it is hard to say beforehand when the economic life of a product stops. For instance, it could be that there is some drastic product innovation by some competitor making the current product obsolete. To model this we impose that the life time of the product, which we denote by T, is stochastic. Taking this into account, we reformulate the objective of the problem as follows:

$$\max_{p,a,u} \mathbf{E}_T \int_0^T e^{-rt}[(p - C(q))D(p, a, q) - a - u]dt,$$

with $\mathbf{E}_T$ denoting the expectation operator with respect to T. Assuming an exponential distribution with parameter $\rho > 0$ for T, like in, e.g., Caulkins et al. (2011), the objective can ultimately be rewritten into

$$\max_{p,a,u} \int_0^\infty e^{-(r+\rho)t}[(p - C(q))D(p, a, q) - a - u]dt.$$

We conclude that now the complete problem looks as follows:

$$\max_{p,a,u} \int_0^\infty e^{-(r+\rho)t}[(p - C(q))D(p, a, q) - a - u]dt. \tag{37}$$

$$\text{subject to } \dot{q} = K(u, q), \text{ with } q(0) = q_0. \tag{38}$$

Comparing the problem (37) and (38) with the problem (21) and (22) shows that they are equivalent except that in the problem (37) and (38) the discount rate has increased by the rate ρ, which makes the firm more myopic. From (35) and (36) we see that the $\dot{q} = 0$ isocline is not affected by a change in the discount rate, while the $\dot{u} = 0$ isocline moves downward if the discount rate becomes larger. This results in less quality investments and a lower long run quality level. We summarize these results as follows:

Proposition 5. *(a) If the end of the life cycle of a product is finite and exponentially distributed with parameter ρ, the problem is equivalent to the original infinite horizon problem with the discount rate, r, being increased by ρ.*

(b) A larger value of r and/or ρ results in less quality investments and a lower long run quality level.

6.3. Comparative statics

After having established how the long run equilibrium is affected by the discount rate r, namely

$$\frac{\partial \hat{q}}{\partial r} < 0, \quad \frac{\partial \hat{u}}{\partial r} < 0,$$

we now investigate the effect of some other parameters. The results are summarized in the following proposition.

Proposition 6. *(a) First we investigate the effect of γ, which measures the effect of quality on the market potential:*

$$\frac{\partial \hat{q}}{\partial \gamma} > 0, \quad \frac{\partial \hat{u}}{\partial \gamma} > 0.$$

(b) Next we consider the effect of θ, which measures the effect of advertising on the market potential:

$$\frac{\partial \hat{q}}{\partial \theta} > 0, \quad \frac{\partial \hat{u}}{\partial \theta} > 0.$$

(c) The next parameter we analyze is β, which is the slope of the demand function:

$$sgn\left(\frac{\partial \hat{q}}{\partial \beta}\right) = sgn\left(\frac{\partial \hat{u}}{\partial \beta}\right) = sgn\left(\frac{\theta^2}{2} - \frac{\gamma}{c} - \beta\right).$$

(d) The next parameter to be investigated is c, being the unit cost that positively depends on quality:

$$\frac{\partial \hat{q}}{\partial c} < 0, \quad \frac{\partial \hat{u}}{\partial c} < 0.$$

(e) If the depreciation rate of quality, δ, increases, the long run optimal level of quality and quality investment react as follows:

$$\frac{\partial \hat{q}}{\partial \delta} < 0, \quad \frac{\partial \hat{u}}{\partial \delta} = -\left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)(4\beta - \theta^2)}\right)^{\frac{1}{\alpha}} (\gamma - \beta c)^2 \times \frac{r + 2\delta + (2\delta - r)\alpha}{\alpha(r + \delta + \alpha\delta)^2(4\beta - \theta^2)}.$$

Proof. See Appendix C.4. □

The first result (a) is that the firm invests more in quality, if the effect of quality on the market potential is bigger. Concerning result (b), the intuition is that advertising becomes more effective, so that the firm advertises more when θ is larger. From our analysis of Step 1, see (46), we have obtained that advertising and quality are complements. Hence the firm also invests more in quality. According to result (c), the effect of the slope of the demand function, β, depends on the sign of $\frac{\theta^2}{2} - \frac{\gamma}{c} - \beta$. This is understandable, since from (29) we get that the same expression determines whether price is increasing with quality or not. In particular we obtain that if

$$\frac{\theta^2}{2} - \frac{\gamma}{c} - \beta > 0,$$

price is decreasing in quality. So, under this relationship we have that an increase in β logically implies that the price is lower and that the firm increases quality, which explains why an increase in β results in a higher long run quality level and investment in quality.

If c increases, according to result (d) the unit cost increases more with quality and therefore the firm is reluctant to invest in quality too much. According to result (e), we first see that if the depreciation rate is bigger, the long run optimal quality level will be lower. It is less clear what happens with the investment level. Two effects can be distinguished. First, if quality depreciated more, quality investments are less profitable, and therefore the firm invests less. Second, if quality depreciates more, the firm has to carry out more replacement investment to keep quality at a reasonable level. Hence, $\frac{\partial \hat{u}}{\partial \delta}$ can have either sign. If $\alpha < 1$, or $2\delta > r$, then the first effect dominates, i.e., $\frac{\partial \hat{u}}{\partial \delta} < 0$.

7. Conclusion

In this research, we proposed a comprehensive, dynamic modeling of price, advertising, and quality. The model is comprehensive, yet with no sacrifice of generality, as it permits general (structural) functional forms for the cost and the relationships between the marketing-mix variables at the demand level. On the basis of an optimal control model, we offer analytic results. First, we generalize the classical condition of Dorfman and Steiner (1954) to a dynamic context. Second, we provide the conditions along which pricing and advertising strategies are aligned with or opposed to quality improvement. These results help better understand the profitable opportunities of the firm. Third, we obtain that quality monotonically converges to a unique and positive steady state. On this time path quality investment can either be increasing or decreasing, depending on how its effectiveness depends on the quality level itself.

Our research on the price–advertising–quality relationship may be expanded into several ways. We could investigate a carry-over role for advertising, which may have a lasting effect through goodwill formation. Further, to better characterize the dynamic behavior of quality cost, quality could be modeled as a control variable, and quality cost as a state variable. This modeling strategy would deepen our understanding of cost dynamics, and its link to pricing and advertising. Also, we may integrate temporal effects, such as fashion effects, in the demand function, which would also depend directly on time. Eventually, profitable markets attract new entrants, raising strategic issues. Consequently, it would be interesting to model competition in our model. Such extensions will allow to discuss the relationship between price, advertising, and quality in a more complete way; they are left for future research.

This article provides significant insights into the choice of the managerial variables over time. The general rule of pricing–advertising–quality relationship, which we present, expands prior results and formulates in a novel way the demand- and supply-sides of the marketing-mix. Such a formulation, in turn, yields a more comprehensive understanding of interplay between price, advertising, and quality. Our theoretical foundation calls for further research to obtain empirical validation.

Appendix A. The rules of Dorfman and Steiner (1954)

We present here the rules of Dorfman and Steiner (1954) as derived from profit maximization. For brevity, second-order conditions are supposed to hold. Recall η_p the (absolute) price elasticity of demand, η_a the advertising elasticity of demand, and η_q the quality elasticity of demand.

A.1. The price–advertising relationship

The demand writes $D = D(p, a)$ and the profit $\pi = [p - C(q)]D(p, a) - a$. The first-order conditions $\pi_p = \pi_a = 0$ jointly impose the rule of price–advertising

$$\frac{a}{pD} = \frac{\eta_a}{\eta_p}. \tag{39}$$

The price–advertising rule is the most well-known rule in Dorfman and Steiner (1954, Section 1). For instance, this rule is recalled in the survey of Bagwell (2007, Section 4.1.1).

A.2. The price–quality relationship

The demand is $D = D(p, q)$ and the profit $\pi = [p - C(q)]D(p, q)$. The first-order conditions $\pi_p = \pi_q = 0$ in conjunction implicate the rule of advertising–quality

$$\frac{dC}{dq} = \frac{\eta_q}{\eta_p}\frac{p}{q}. \tag{40}$$

The price–quality rule originates in Dorfman and Steiner (1954, Section 2). It appears later in Teng and Thompson (1996) and Lin (2008).

A.3. The advertising–quality relationship

The demand is $D = D(a, q)$ and the profit $\pi = [p - C(q)]D(a, q) - a$, with $p > C$ a constant. The first-order conditions $\pi_a = \pi_q = 0$ together dictate the rule of advertising–quality

$$\frac{dC}{dq}D = \frac{\eta_q}{\eta_a}\frac{a}{q}. \tag{41}$$

The advertising–quality rule is not explicitly written in Dorfman and Steiner (1954, Section 3). Also and to the best of our knowledge, this rule has not been explicitly proposed in the literature. But the rule can be directly derived from the framework of Dorfman and Steiner (1954) as above.

A.4. Comparison between Proposition 1 and the rules in Dorfman and Steiner (1954)

For the sake of completeness, we compare Proposition 1 against Dorfman and Steiner (1954). The famous condition of Dorfman and Steiner (1954) in (39) characterizes the optimal policies of price and advertising. More precisely, this condition describes the structural properties of these policies, linking the relative advertising spending to the price and advertising elasticities of demand. This condition has to hold for any couple of optimal price and advertising policies, constituting a simple benchmark if the elasticities hold constant. Also, numerous empirical studies have confirmed the condition Dorfman and Steiner (1954) in different contexts. Bagwell (2007) notes that this condition, though constraining, still allows for freedom in the casual relationships between price and advertising.

Dorfman and Steiner's (1954)) rules (39)–(41) originate from a static setting, whereas our rules (17a)–(17c) come from a dynamic setting. Simple comparison shows that their rules (with strict equalities for any t over $[0, \infty)$) are a special case of our rules (with weak inequalities over $[t, \infty)$). More precisely, the equality of the price–advertising rule (39) is robust in our dynamic setting, but the equalities of the price–quality and advertising–quality rules (40) and (41) do not hold strictly in our setting.

Two reasons explain the difference between the two sets of rules. First, in our setting, quality is a result of quality investment and there is no restrictive optimality condition with respect to quality. In contrast, in Dorfman and Steiner (1954), quality is directly chosen by the firm and there is an optimality condition with respect to quality. Our modeling thus benefits from more freedom. Second, our modeling refers to a dynamic framework where the firm invests to increase future quality that will drive future profit; quality has a future value for the firm. In contrast, their modeling is static and the firm has only interest in current profit; quality has no future value for the firm. Though, our modeling still enables the equalities (40) and (41) to apply. In fact, these inequalities hold (together) according to (16b) and (16c) if $\lambda = 0$, that is if the future level of quality is of no interest for the firm maximizing the current intertemporal profit. In other words, if the firm behaves in a static way in the sense of considering only current profit and disregarding future profit tied to future quality, then (17b) and (17c) reduce to (40) and (41). In a nutshell, rules (17a)–(17c) differ from (39) to (41) because (1) quality results from investment and (2) future quality matters.

Appendix B. Mathematical details of Sections 4 and 5

B1. Proof of Proposition 1

Substituting the elasticity notation in (16a) and rearranging yields the necessary and sufficient condition (17a). Recalling from (15) that future quality increases future profit, that is $\lambda(t) \geq 0$, together with the value of $\lambda(t)$ measured in (16b) and (16c) implicates the sufficient conditions (17b) and (17c).

B2. Proof of Proposition 2

Differentiate with respect to time the condition of Dorfman–Steiner as expressed in (16a).

B3. Proof of Proposition 3

Solve Eqs. (20a) and (20b) with Cramer's rule.

Appendix C. Mathematical details of Section 6

C.1. Proof of Proposition 4

Solving the Step 1 problem gives the first order conditions

$$\pi_p = \gamma q + \theta\sqrt{a} - 2\beta p + \beta c q = 0, \tag{42}$$

$$\pi_a = \theta \frac{p - cq}{2\sqrt{a}} - 1 = 0. \tag{43}$$

It is straightforward to check that the second order conditions are satisfied (so that we indeed have developed conditions for a maximum) if

$$\pi_{pp}\pi_{aa} - \pi_{ap}^2 \geq 0,$$

which holds if and only if (26) holds. Otherwise, concavity is violated and unbounded solutions might emerge.

We can solve (43) and (42) to obtain

$$p(q) = \frac{\left(2\beta - \theta^2\right)c + 2\gamma}{4\beta - \theta^2} q, \tag{44}$$

$$a(q) = \left(\frac{\theta(\gamma - \beta c)}{4\beta - \theta^2}\right)^2 q^2. \tag{45}$$

From these expressions we get the dependence of price and advertising on quality,

$$p'(q) = \frac{\left(2\beta - \theta^2\right)c + 2\gamma}{4\beta - \theta^2},$$

$$a'(q) = \frac{2\theta^2}{\left(4\beta - \theta^2\right)^2}(\gamma - \beta c)^2 q > 0. \tag{46}$$

We conclude that price increases in quality, if (29) holds, whereas it always holds that the firm advertises more if the product quality is higher.

Substitution of (44) and (45) into (23) learns that demand is proportional to quality, i.e.

$$D = \frac{2\beta}{4\beta - \theta^2}(\gamma - \beta c)q,$$

and we conclude that positive demand only arises if

$$\gamma - \beta c > 0, \tag{47}$$

which completes the proof.

C.2. Derivation of the canonical system of the Step 2 problem

To solve the Step 2 optimal control problem (31) and (32), we set up the Hamiltonian

$$H = \frac{(\gamma - \beta c)^2}{4\beta - \theta^2} q^2 - u + \lambda\left(q^{-\alpha}\sqrt{u} - \delta q\right).$$

Maximizing H with respect to u gives

$$\lambda = 2q^{\alpha}\sqrt{u}, \tag{48}$$

whereas the co-state equation satisfies

$$\dot{\lambda} = r\lambda - 2\frac{(\gamma - \beta c)^2}{4\beta - \theta^2} q - \lambda K_q. \tag{49}$$

We now establish the optimal trajectories in a state-control phase diagram (q, u). To do so, we first differentiate (48) with respect to time, which leads to

$$\dot{\lambda} = 2\alpha q^{-1} u - 2\delta\alpha q^{\alpha}\sqrt{u} + q^{\alpha}\frac{1}{\sqrt{u}}\dot{u}.$$

This can be combined with (49) and (48) to obtain the $\dot{u}$ equation:

$$\dot{u} = 2u(r + \delta + \alpha\delta) - 2\frac{(\gamma - \beta c)^2}{4\beta - \theta^2} q^{1-\alpha}\sqrt{u}. \tag{50}$$

C.3. Stability analysis of the interior steady state

We investigate the stability properties of the interior steady state. The Jacobian of the canonical system

$$\dot{q} = q^{-\alpha}\sqrt{u} - \delta q,$$

$$\dot{u} = 2u(r + \delta + \alpha\delta) - q^{1-\alpha}\frac{2(\gamma - \beta c)^2}{4\beta - \theta^2}\sqrt{u},$$

is

$$J = \begin{bmatrix} -\alpha q^{-1-\alpha}\sqrt{u} - \delta & \frac{1}{2\sqrt{u}} q^{-\alpha} \\ -(1-\alpha)q^{-\alpha}\frac{2(\gamma - \beta c)^2}{4\beta - \theta^2}\sqrt{u} & 2(r + \delta + \alpha\delta) - q^{1-\alpha}\frac{(\gamma - \beta c)^2}{\left(4\beta - \theta^2\right)\sqrt{u}} \end{bmatrix}, \tag{51}$$

with the determinant

$$\det J = -2\alpha(r + \delta + \alpha\delta)q^{-1-\alpha}\sqrt{u} - 2\delta(r + \delta + \alpha\delta) + q^{1-\alpha}\frac{\delta(\gamma - \beta c)^2}{\left(4\beta - \theta^2\right)\sqrt{u}} + \frac{(\gamma - \beta c)^2}{4\beta - \theta^2} q^{-2\alpha}.$$

For the positive steady state (33) and (34), we have

$$\det J = -2\delta\alpha(r + \delta + \alpha\delta) < 0.$$

Hence, the interior steady state is a saddle point.

C.4. Proof of Proposition 6

(a) First we investigate the effect of γ:

$$\frac{\partial\hat{q}}{\partial\gamma} = \frac{\gamma - \beta c}{\alpha\left(\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)\right)} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^{\frac{1}{2\alpha} - 1} > 0,$$

$$\frac{\partial\hat{u}}{\partial\gamma} = \frac{2(1 + \alpha)(\gamma - \beta c)\delta^2}{\alpha\left(\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)\right)} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^{\frac{1}{\alpha}} > 0,$$

where the sign follows from (47).

(b) Next we consider the effect of θ:

$$\frac{\partial\hat{q}}{\partial\theta} = \frac{(\gamma - \beta c)^2\theta}{\alpha\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)^2} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^{\frac{1}{2\alpha} - 1} > 0,$$

$$\frac{\partial\hat{u}}{\partial\theta} = \frac{2(1 + \alpha)\delta^2(\gamma - \beta c)^2\theta}{\alpha\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)^2} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^{\frac{1}{\alpha}} > 0.$$

(c) The next parameter we analyze is β:

$$\frac{\partial\hat{q}}{\partial\beta} = \frac{2c}{\alpha}\frac{(\gamma - \beta c)}{\delta(r + \delta + \alpha\delta)}\frac{\frac{\theta^2}{2} - \frac{\gamma}{c} - \beta}{\left(4\beta - \theta^2\right)^2} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r + \delta + \alpha\delta)\left(4\beta - \theta^2\right)}\right)^{\frac{1}{2\alpha} - 1},$$

$$\frac{\partial \hat{u}}{\partial \beta} = 4c\delta^2 \frac{1+\alpha}{\alpha} \frac{(\gamma - \beta c)}{\delta(r+\delta+\alpha\delta)} \frac{\frac{\theta^2}{2} - \frac{\gamma}{c} - \beta}{\left(4\beta - \theta^2\right)^2} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r+\delta+\alpha\delta)\left(4\beta - \theta^2\right)} \right)^{\frac{1}{\alpha}},$$

from which we obtain that it depends on the sign of $\frac{\theta^2}{2} - \frac{\gamma}{c} - \beta$ whether an increase of β will lead to higher quality investments or not.

(d) The next parameter to be investigated is c:

$$\frac{\partial \hat{q}}{\partial c} = -\frac{\beta(\gamma - \beta c)}{\alpha\delta(r+\delta+\alpha\delta)\left(4\beta - \theta^2\right)} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r+\delta+\alpha\delta)\left(4\beta - \theta^2\right)} \right)^{\frac{1}{2\alpha}-1} < 0,$$

$$\frac{\partial \hat{u}}{\partial c} = -\frac{2(1+\alpha)\beta(\gamma - \beta c)\delta}{\alpha(r+\delta+\alpha\delta)\left(4\beta - \theta^2\right)} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r+\delta+\alpha\delta)\left(4\beta - \theta^2\right)} \right)^{\frac{1}{\alpha}} < 0.$$

(e) Finally, we consider δ:

$$\frac{\partial \hat{q}}{\partial \delta} = -\frac{(\gamma - \beta c)^2(r+2\delta+2\alpha\delta)}{2\alpha\left(\delta(r+\delta+\alpha\delta)\right)^2\left(4\beta - \theta^2\right)} \times \left(\frac{(\gamma - \beta c)^2}{\delta(r+\delta+\alpha\delta)\left(4\beta - \theta^2\right)} \right)^{\frac{1}{2\alpha}-1} < 0,$$

$$\frac{\partial \hat{u}}{\partial \delta} = -\left(\frac{(\gamma - \beta c)^2}{\delta(r+\delta+\alpha\delta)\left(4\beta - \theta^2\right)} \right)^{\frac{1}{\alpha}} (\gamma - \beta c)^2 \times \frac{r+2\delta+(2\delta - r)\alpha}{\alpha(r+\delta+\alpha\delta)^2\left(4\beta - \theta^2\right)}.$$

This completes the proof.

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